Physics 798C
Superconductivity
Spring 2024
Homework 4
Due Thursday 29 February, 2024

1. **Collapsing products of non-commuting operators**
   When checking the normalization of the BCS ground state wavefunction:

$$\left|\Psi_{G,BCS}\right\rangle = \prod_{k=k_1}^{k=k_M}\left(u_k + v_k c^{+}_{\vec{k}\uparrow} c^{+}_{-\vec{k}\downarrow}\right)\left|0\right\rangle$$

, where the product is over all momentum states from k_1 to k_M, one has to consider a double product over all momentum states when constructing $\left\langle\Psi_{G,BCS}\middle|\Psi_{G,BCS}\right\rangle$. Show that this double product can be safely collapsed into a single product by carefully considering the anti-commuting properties of the Fermionic operators. We are interested in demonstrating this collapse in general, and use the normalization calculation simply to illustrate the point.

2. **BCS Variational Calculation**
   Starting from the BCS pairing Hamiltonian in terms of the $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, work through the variational calculation and derive the final results for $u_{\mathbf{k}}$, $v_{\mathbf{k}}$, and the zero temperature gap Δ.

In other words, start from this equation:

$$\langle\Psi_G|\mathcal{H} - \mu\, N_{op}|\Psi_G\rangle = 2\sum_k \xi_k\, |v_k|^2 + \sum_{kl} V_{kl}\, u_k\, v_k^*\, u_l^*\, v_l$$

… and arrive at these equations:

$u_k^2 = \frac{1}{2}\left[1 + \frac{\epsilon_k - \mu}{\sqrt{\Delta^2 + (\epsilon_k - \mu)^2}}\right]$ and $v_k^2 = \frac{1}{2}\left[1 - \frac{\epsilon_k - \mu}{\sqrt{\Delta^2 + (\epsilon_k - \mu)^2}}\right]$